MISSING PROOFS SECTION 2.2 AND SOME OTHER FACTS

Example 1. *Assume that A is an $m \times n$ matrix and B is an $n \times p$ matrix. Prove $(AB)^T = B^T A^T$.*

Proof. Let $C = (AB)^T = [c_ij]$. We denote the (i, j)-th entry of A and B by a^T_{ij} and b^T_{ij} respectively. Recall

$$\begin{aligned} c_{ij} &= (i, j)\text{-th entry of } AB = (AB)_{ij} = \sum_{i=1}^{n} a_{jk} b_{ki} \\ &= \sum_{i=1}^{n} a^T_{kj} b^T_{ik} = \sum_{i=1}^{n} b^T_{ik} a^T_{kj} \\ &= (i, j)\text{-th entry of } B^T A^T. \end{aligned}$$

□

Example 2. *Assume that A, B are $m \times n$ matrices and C is $n \times p$. Prove that $(A + B)C = AC + BC$.*

Proof. Let $(A + B)C = D = [d_{ij}]$. Then

$$\begin{aligned} d_{ij} &= \sum_{k=1}^{n} (A + B)_{ik} c_{kj} = \sum_{k=1}^{n} (a_{ik} + b_{ik}) c_{kj} \\ &= \sum_{k=1}^{n} a_{ik} c_{kj} + \sum_{k=1}^{n} b_{ik} c_{kj} = (AC)_{ij} + (BC)_{ij} \\ &= (AC + BC)_{ij}. \end{aligned}$$

By the definition of matrix equality, this implies $(A + B)C = AC + BC$. □

Example 3. *Prove that the product of any two upper(lower) triangular matrices is again a(n) upper(lower) triangular matrix.*

Remark 4. *For any $n \times n$ matrix A, A is upper triangular if and only if*

$$a_{ij} = 0, \quad \text{for all } i > j,$$

and A is lower triangular if and only if

$$a_{ij} = 0, \quad \text{for all } i < j.$$

Proof. (of Example 2) We will prove this assertion for upper triangular matrices only. The proof for lower triangular case is exactly the same.

Suppose that $A_{n\times n}, B_{n\times n}$ are both upper triangular. Let $AB = C = [c_{ij}]$. Then

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

We consider only c_{ij} with

$$i > j. \tag{1}$$

Case I: $k > j$

In this case, $b_{kj} = 0$. So $a_{ik}b_{kj} = 0$.

Case II: $k \leq j$.

In this case, by (1)

$$i > j \geq k,$$

which implies $a_{ik} = 0$. Thus $a_{ik}b_{kj} = 0$.

In both cases, we obtain

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = 0, \quad \text{for } i > j.$$

We thus infer from Remark 4 that C is upper triangular. □